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Probabilistic design and quality control in probabilistic strength of materials

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Abstract

A numerical simulation method is used here for the design and quality control of a material subject to normal gradual stress σ or a cyclic stress σ, having fixed cumulative probability F and the number of cycles l; capable of achieving a given mechanical property such as yield point, elastic limit stress, fracture strength, etc., as well as the admissible tolerance δF the presence of such property is to be accepted with. With F and δF, the stress of the design σ_C can be determined, as well as the variations δm, $\delta\sigma_0$ and $\delta\sigma_L$ of Weibull's parameters m, σ_0 and σ_L, respectively, that the tolerance δF admits. When cyclic stresses arise, other parameters must be introduced, k_1, k_2 and p, which produce variations δk_1, δk_2, and δp, respectively. The determination of the necessary number of samples to be tested in order to carry out the quality control of the material, with a given probability of effectiveness, is obtained with variations δm, $\delta\sigma_0$, $\delta\sigma_L$, δk_1, δk_2 and δp, with the parameter dispersion estimated by numerical simulation, and with the help of a property deduced from the Fischer's matrix to obtain the parameter dispersion.

Keywords: Numerical methods; Mechanical property; Stress; Cyclic

1. Introduction

The use of probabilistic strength of materials (Weibull, 1939; Kittl and Díaz, 1988) had been directed, in general, to determine the Weibull parameters of Weibull's cumulative probability function for some material subjected to both constant and variable stress fields for mechanical components in service or for isolated samples manufactured to obtain such parameters. This probabilistic approach had been used on diverse materials (Kittl and Díaz, 1988) subjected to gradual stress. When fatigue is included the problem is more complex due to the singular characteristics of the fatigue process, so it needs the introduction of other

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parameters that consider such complexity. The mechanical design and quality control for components subjected to gradual or cyclic stress, treated from a probabilistic point of view, requires of the knowledge of the dispersions and the variations of the different parameters used to characterize the load process for a certain level of cumulative probability of occurrence of some mechanical property, in order to define a design stress value, including an admissible tolerance for such probability.

The main objective of this work is to obtain, through a numerical simulation method, the required number of samples to be tested to define the design and quality control for some material subjected to traction, including gradual and cyclic stress, that satisfy a fixed cumulative probability with a corresponding admissible tolerance for them.

2. Gradual stress

According to Weibull's theory (Kittl and Díaz, 1988; Kittl et al., 2001) the cumulative probability of occurrence of some mechanical property $F(\sigma)$, for a material subjected to uniaxial stress field, is given by the following expression:

$$F(\sigma) = 1 - \exp\left\{ -\frac{V}{V_0}\left(\frac{\sigma - \sigma_L}{\sigma_0}\right)^m \right\} \tag{1}$$

where σ is the stress at which some mechanical property is verified, such as yield point, elastic limit stress, fracture strength; V is the volume of the material, V_0 is the unit of volume, σ_L is one of Weibull's parameters and is the limit of stress bellow which the property will not occur; and the other Weibull's parameters are m and σ_0 which depend on the manufacture process of the material. As all the samples are equal, $V = V_0$.

Once cumulative probability F is fixed design stress, σ_C, is obtained from Eq. (1). In addition, along with cumulative probability F there is an admissible tolerance for such probability, δF, associated to variations δm, $\delta\sigma_0$ and $\delta\sigma_L$ of Weibull's parameters m, σ_0 and σ_L, respectively. Such variations can be obtained from F and δF using the following equations:

$$F = 1 - \exp\left\{ -\left(\frac{\sigma_C - \sigma_L}{\sigma_0}\right)^m \right\} \tag{2a}$$

$$F + \delta F = 1 - \exp\left\{ -\left(\frac{\sigma_C - \sigma_L}{\sigma_0}\right)^{m+\delta m} \right\} \tag{2b}$$

$$F + \delta F = 1 - \exp\left\{ -\left(\frac{\sigma_C - \sigma_L}{\sigma_0 + \delta\sigma_0}\right)^m \right\} \tag{2c}$$

$$F + \delta F = 1 - \exp\left\{ -\left(\frac{\sigma_C - (\sigma_L + \delta\sigma_L)}{\sigma_0}\right)^m \right\} \tag{2d}$$

Provided that any part of the material is calculated using the same σ_C, the variations producing an admissible tolerance δF must be originated in Weibull's parameters, that is to say, they come from the material (see Fig. 1).

For the same value of σ_C, when the material, characterized by its respective Weibull's parameters, is changed from (m, σ_L, σ_0) to $(m + \delta m, \sigma_0 + \delta\sigma_0, \sigma_L + \delta\sigma_L)$, cumulative probability F is changed to $F + \delta F$.

With a close approximation, Eqs. (2a) and (2b) determine δm, Eqs. (2a) and (2c) determine $\delta\sigma_0$, and Eqs. (2a) and (2d) determine $\delta\sigma_L$. It is easily deduced from Fischer's matrix (Kittl and Díaz, 1988) that the dispersion of Weibull's parameters Δm, $\Delta\sigma_L$ and $\Delta\sigma_0$ are given by equations:

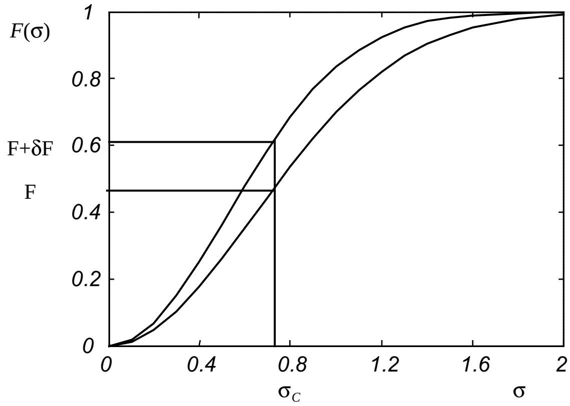

Fig. 1. Diagram of cumulative probability.

$$\begin{aligned}\Delta m &= \frac{1}{\sqrt{N}} f(m, \sigma_0, \sigma_L) \\ \Delta\sigma_0 &= \frac{1}{\sqrt{N}} g(m, \sigma_0, \sigma_L) \\ \Delta\sigma_L &= \frac{1}{\sqrt{N}} h(m, \sigma_0, \sigma_L)\end{aligned} \tag{3}$$

where N is the number of samples to be tested to obtain Δm, $\Delta\sigma_0$ and $\Delta\sigma_L$; functions f, g and h depend only of m, σ_0 and σ_L, they are constants that may be deduced from a particular case. When the material is subjected to a variable stress field the use of Fisher's matrix is more complex and is not dealt with here (Díaz et al., 2002). Thus, in this situation, a numerical simulation method to obtain the dispersion of the Weibull's parameters has been developed starting from values m, σ_0 and σ_L, determined through experimental values. Normally the resulting values of m, σ_0 and σ_L are obtained using $N = 30$ testing samples and are an outcome of obtaining the maximum correlation between independent variable $\ln(\sigma - \sigma_L^i)$ and $\ln[\ln(1/(1-F))]$, where σ_L^i are several values of σ_L that are taken from around the smallest value of σ. From Eq. (1) we obtain:

$$\ln \xi(\sigma) = \ln\left[\ln\left(\frac{1}{1-F}\right)\right] = m\ln(\sigma - \sigma_L) - m\ln\sigma_0 \tag{4}$$

Thus, $\ln \xi(\sigma)$ is a linear function of $\ln(\sigma - \sigma_L)$. In the Weibull diagram, $\ln \xi(\sigma)$ versus $\ln \sigma$, when $\sigma = \sigma_L$ then $\ln \xi(\sigma) = -\infty$, which is a good way for obtaining the approximate value of σ_L. In order to improve σ_L several values are chosen in the neighborhood of that first σ_L value obtained and the best correlation with better chi-square value, χ^2, allows for the determination of such parameter.

In order to show how the numerical simulation method works Weibull's parameters obtained from an AISI 1020 steel (Díaz et al., 1999) subjected to traction test were used. Such steel was elected due to the fact that it is fairly ductile and is usually employed in the construction of structures. In the case of this steel, the following values can be considered: $m = 14$, $\sigma_0 = 53$ MPa and $\sigma_L = 400$ MPa.

The numerical simulation method used here does not require special software, it only needs a routine to generate random numbers, and the following numerical determinations are easy. The method starts with the knowledge of Weibull's parameters m, σ_0 and σ_L from an experimental test for some material subjected to traction. Therefore we know the cumulative probability function given by Eq. (1). Then, using a random number generation function we generated 30 random numbers and each one was replaced on the left side of Eq. (1) in order to determine the respective stress value. This way we have a set of 30 simulated stress values. After that, the set is put in a rising order. With the set sorted out, the cumulative probability is evaluated using an estimator. And a new Weibull function is obtained. Finally, this process allows us to estimate new values for Weibull's parameters m, σ_0 and σ_L. Many iterations can be made in order to obtain other values for Weibull's parameters. The computation scheme is:

$$\{0 \leqslant \lambda_i^j \leqslant 1\} \rightarrow \{\cdots \sigma_i^j \leqslant \sigma_{i+1}^j \leqslant \cdots\} \leftrightarrow \left\{F_i^j = \frac{i - \frac{1}{2}}{N}\right\} \rightarrow \{m^j, \sigma_0^j, \sigma_L^j\} \quad j = 1, \ldots, 100; \quad i = 1, \ldots, N = 30 \tag{5}$$

where λ_i^j are aleatory numbers, σ_i^j are simulated stress values, F_i^j are cumulative probability estimators, N is the total number of samples tested, and m^j, σ_0^j and σ_L^j are the Weibull parameters obtained by linear regression using σ_L^j from approximations with a maximum correlation coefficient. One hundred simulations of 30 testing samples were carried out and the cumulative probabilities $F(m)$, $F(\sigma_0)$ and $F(\sigma_L)$ were obtained using the respective 100 values for each Weilbull parameter, as showed in Figs. 2–4. In order to calculate Δm, $\Delta\sigma_0$ and $\Delta\sigma_L$ all the values that depart from a normal distribution were eliminated, thus avoiding possible calculation errors. This can be seen in the isolated points on the right in Fig. 2 and on the left in Fig. 4.

In this case $j = 1, \ldots, 100$, $i = 1, \ldots, 30$ and the dispersion values of Weibull's parameters were: $\Delta m = 5.25$, $\Delta\sigma_0 = 10.6$ MPa and $\Delta\sigma_L = 8.06$ MPa. With the help of Eq. (3) we may evaluate f, g and h:

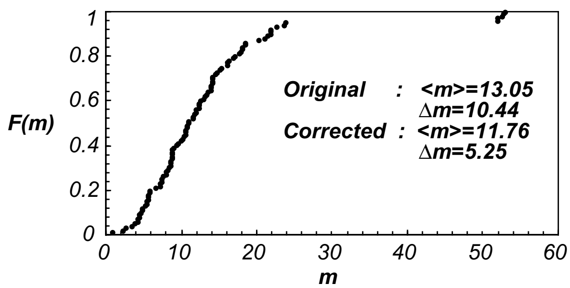

Fig. 2. Cumulative probability of Weibull's parameter m.

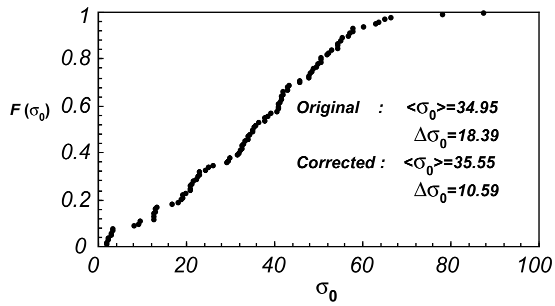

Fig. 3. Cumulative probability of Weibull's parameter σ_0.

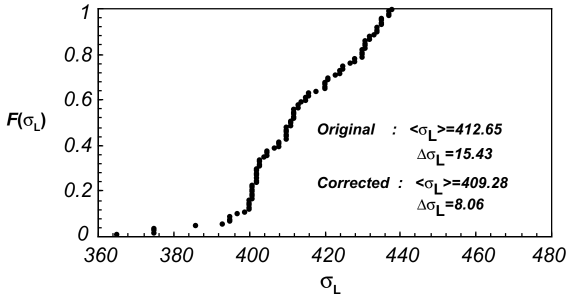

Fig. 4. Cumulative probability of Weibull's parameter σ_L.

$$\begin{aligned} f &= \sqrt{30}\Delta m = 28.8 \\ g &= \sqrt{30}\Delta\sigma_0 = 58.1 \text{ MPa} \\ h &= \sqrt{30}\Delta\sigma_L = 44.2 \text{ MPa} \end{aligned} \tag{6}$$

For another hand Eq. (2) allows one to determine the expressions for the variations of Weibull's parameters δm, $\delta\sigma_0$ and $\delta\sigma_L$ and then we can find the number of testing samples required in a quality control in order to get a dispersion compatible with the fixed values for the cumulative probability and its respective admissible tolerance, F and δF. Such fixed values for F and δF depend on design conditions and are a variable managed by the designer. The following equations shows how to determine the variations of Weibull's parameters:

$$\begin{aligned} \delta m &= \left\{ \frac{\ln\left[\ln\left(\frac{1}{1-(F+\delta F)}\right)\right]}{\ln\left[\ln\left(\frac{1}{1-F}\right)\right]} - 1 \right\} m \\ \delta\sigma_0 &= \left\{ \left[\frac{\ln\left(\frac{1}{1-F}\right)}{\ln\frac{1}{1-(F+\delta F)}} \right]^{1/m} - 1 \right\} \sigma_0 \\ \delta\sigma_L &= \left\{ \frac{\sigma_C - \sigma_0\left[\ln\left(\frac{1}{1-(F+\delta F)}\right)\right]^{1/m}}{\sigma_C - \sigma_0\left[\ln\left(\frac{1}{1-F}\right)\right]^{1/m}} - 1 \right\} \sigma_L \\ \sigma_C &= \sigma_0\left[\ln\left(\frac{1}{1-F}\right)\right]^{1/m} + \sigma_L \end{aligned} \tag{7}$$

where σ_C is the design stress value. Now, using Eq. (3), i.e., the relations obtained from Fischer's matrix, that allows us to obtain the dispersions of Weibull's parameters, with Eq. (7) we can determine the number of samples to be tested through the following equations:

$$Nm = \frac{f^2}{\delta m^2}, \quad N\sigma_0 = \frac{g^2}{\delta\sigma_0^2}, \quad N\sigma_L = \frac{h^2}{\delta\sigma_L^2} \tag{8}$$

where Nm, $N\sigma_0$ and $N\sigma_L$ are the number of samples to be tested to determine Weibull's parameters m, σ_0 and σ_L, respectively, compatible with the fixed cumulative probability and with its respective admissible tolerance. Eqs. (7) and (8) provide for a criterion to expect that approximately 70% of σ values are within a range of acceptance. If we take 2δ instead of δ, we obtain approximately 90% of the values of σ within the range of acceptance. The highest value of the numbers of samples to be tested, given by Eq. (8), must be chosen. In the case of quality control for some material when a set of material samples to be used is available, and N samples are tested, where N is the largest value obtained from Eq. (8), parameters m, σ_0 and σ_L can be determined, and for design stress value, σ_C, if the new cumulative probability, obtained from Eq. (7) is higher than the previous $F + \delta F$, due to new Weibull parameter values then the set of samples must be rejected or a new σ_C must be obtained.

3. Determination of Weibulls parameters by the moment method in gradual stress

From Weibull's cumulative probability function given by Eq. (1), it is easy to show that (Kittl and Díaz, 1988; Kittl et al., 2001):

$$\frac{\Delta\sigma}{\overline{\sigma} - \sigma_L} = \frac{\left[\Gamma\left(1 + \frac{2}{m}\right) - \Gamma^2\left(1 + \frac{1}{m}\right)\right]^{1/2}}{\Gamma\left(1 + \frac{1}{m}\right)} \tag{9}$$

where $\overline{\sigma}$ is the mean value, $\Delta\sigma$ is the dispersion and Γ is Euler's gamma function. The mean value and its dispersion are given by the following equations:

$$\begin{aligned} \overline{\sigma} &= \frac{1}{N}\sum_{i=1}^{N}\sigma_i \\ \Delta\sigma &= \sqrt{\frac{1}{N-1}\cdot\sum_{i=1}^{N}(\sigma_i - \overline{\sigma})^2} \end{aligned} \tag{10}$$

both are estimated for a series of testing experimental values σ_i and Weibull's parameter m is found by using Eq. (9). In order to introduce σ_L in Eq. (9), trials are made with different values within the vicinity of a σ_L obtained with a certain approximation from a Weibull diagram $\ln\xi(\sigma) = \ln[\ln(\frac{1}{1-F})]$ versus $\ln\sigma$ as shown in Fig. 5. Then, for the value chosen for σ_L the chi-square value, χ^2, must be minimized (see Fig. 6). The other Weibull parameter σ_0 is obtained from the following expression:

$$\sigma_0 = \frac{\overline{\sigma}}{\Gamma\left(1 + \frac{1}{m}\right)} \tag{11}$$

When σ_L is ignored we proceed by using Eq. (9) with $\sigma_L = 0$.

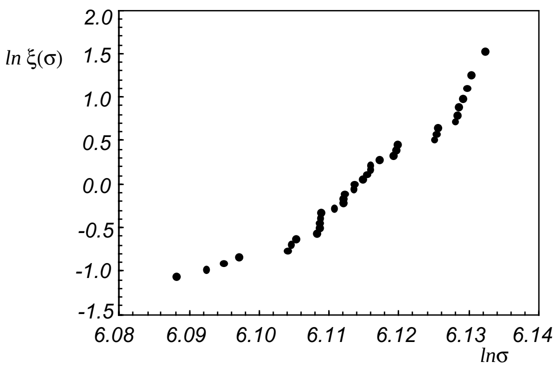

Fig. 5. Weibull's diagram of a series of values of σ, corresponding to a traction test for an AISI 1020 steel.

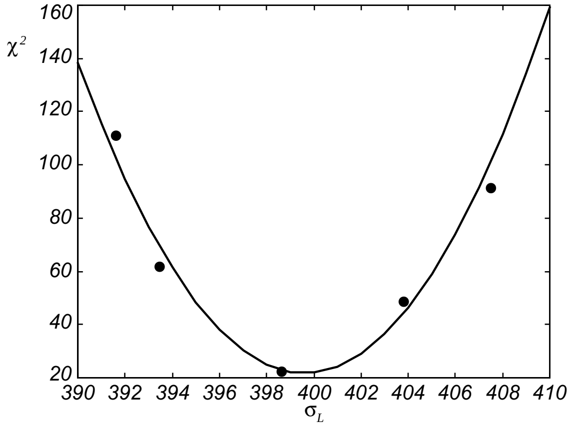

Fig. 6. Determination of σ_L by obtaining the minimum value of chi-square, χ^2.

4. Dispersion of Weibulls parameters and estimation of the number of samples to be tested for gradual stress

In the case of the steel employed here with the value of Weibull's parameters m, σ_0 and σ_L, the numerical simulation process described in scheme (5) was used to generate 10 simulated test with $N = 30$ samples in order to obtain the mean values of Weibull's parameters and their dispersions when the limit stress σ_L is both different from zero and equal to zero. Their respective values are shown in Table 1.

F being the level of cumulative design probability, and δF its respective tolerance, the variation of parameters δm, $\delta\sigma_0$ and $\delta\sigma_L$, is obtained from Eq. (2). Table 2 shows the respective Weibull's parameter

Table 1
Weibull's parameters m, σ_0 and σ_L, their mean values and their dispersions for AISI 1020 steel samples, subjected to traction test, randomly generated by numerical simulation process, considering limit stress both $\sigma_L \neq 0$ and $\sigma_L = 0$

Limit stress σ_L	Weibull's parameters	Mean value	Dispersions
$\sigma_L \neq 0$	m	15.6	$\Delta m = 1$
	σ_0	54 MPa	$\Delta\sigma_0 = 0.5$ MPa
	σ_L	400 MPa	$\Delta\sigma_L = 0$ MPa
$\sigma_L = 0$	m	135.3	$\Delta m = 21$
	σ_0	453 MPa	$\Delta\sigma_0 = 0.83$ MPa

Table 2
Weibull's parameters deviations δm, $\delta\sigma_0$ and δ_L and design stress values for AISI 1020 steel, subjected to traction test, with a fixed cumulative probability F and admissible tolerance δF, with $F = 10^{-7}$ and $F + \delta F = 10^{-6}$, and considering limit stress both $\sigma_L \neq 0$ and $\sigma_L = 0$

Limit stress σ_L	Weibull's parameters	Variations parameters	Stress of design σ_C
$\sigma_L \neq 0$	m	$\delta m = -2$	417 MPa
	σ_0	$\delta\sigma_0 = -8$ MPa	
	σ_L	$\delta\sigma_L = -3$ MPa	
$\sigma_L = 0$	m	$\delta m = -19.3$	402 MPa
	σ_0	$\delta\sigma_0 = -7.64$ MPa	

Table 3
Number of samples to be tested to estimate Weibull's parameters m, σ_0 and σ_L with a fixed cumulative probability F and admissible tolerance δF, with $F = 10^{-7}$ and $F + \delta F = 10^{-6}$, for AISI 1020 steel samples, subjected to traction test, randomly generated by numerical simulation process, considering limit stress both $\sigma_L \neq 0$ and $\sigma_L = 0$

Limit stress σ_L	Number of samples to estimate the corresponding Weibull's parameters		
	Nm	$N\sigma_0$	$N\sigma_L$
$\sigma_L \neq 0$	207	53	217
$\sigma_L = 0$	36	0.35	–

variations obtained when limit stress σ_L is both different from zero and equal to zero in the case $F = 10^{-7}$ and $F + \delta F = 10^{-6}$.

All dispersions in each Weibull parameter may be increased by a constant factor, but in order to simplify let a same factor μ for each one be, where μ is a real number. Then, we can generalize the use of Fisher's matrix property (Kittl and Díaz, 1988), given by Eq. (3), to obtain the number of samples to be tested to determine the Weibull parameters compatible with the fixed cumulative probability and with its respective admissible tolerance. Therefore, considering Eqs. (3) and (8), it gives:

$$\begin{aligned} \mu\Delta m &= \frac{1}{\sqrt{N}} f(m, \sigma_0, \sigma_L); \quad Nm = \frac{f^2}{\delta^2 m} = \frac{\mu^2 \cdot \Delta^2 m \cdot N}{\delta^2 m} \\ \mu\Delta\sigma_0 &= \frac{1}{\sqrt{N}} g(m, \sigma_0, \sigma_L); \quad N\sigma_0 = \frac{g^2}{\delta^2 \sigma_0} = \frac{\mu^2 \cdot \Delta^2 \sigma_0 \cdot N}{\delta^2 \sigma_0} \\ \mu\Delta\sigma_L &= \frac{1}{\sqrt{N}} h(m, \sigma_0, \sigma_L); \quad N\sigma_L = \frac{h^2}{\delta^2 \sigma_L} = \frac{\mu^2 \cdot \Delta^2 \sigma_L \cdot N}{\delta^2 \sigma_L} \end{aligned} \tag{12}$$

When $\mu = 1$ the reliability is approximately equal to 70%. The comments below Eq. (8) can also be applied here. Table 3 shows such number of test samples in order to obtain Weibull's parameters with a fixed cumulative probability F and with a fixed admissible tolerance δF for a determined design stress value σ_C, when the limit stress σ_L is both different from zero and equal to zero.

The difference between design stress values σ_C when the limit stress σ_L is both different from zero and equal to zero is: (417 − 402 MPa)/402 MPa ≈ 3% i.e. a very small value. Thus, the error would not be too large and the design would be safer if value $\sigma_L = 0$ is adopted.

5. Cyclic stress

In the case of the material being subjected to cyclic loads the phenomenon can be studied using a probabilistic approach too and taking into account different mechanical properties. Fig. 7 shows the evolution of one of these mechanical properties, such as yield point, elastic limit stress, fracture strength, with the application of a gradual load. We will assign only one cycle $l = 10^0$ to this load, where l is the number of load cycles. When more cycles are applied, $l = 10^1, l = 10^2, \ldots, l = 10^7$, the curve of the cumulative probability by cyclic stress σ and number of cycles l, $F(\sigma, l)$, takes up values starting from a limit stress σ_L, that we shall consider independent from l due to the mechanism of fatigue by cyclic loads.

For a given σ, the way in which l varies is shown in Fig. 8, where it can be presumed that there is an evolution similar to one of Weibull's function for $F(\sigma, l = k)$.

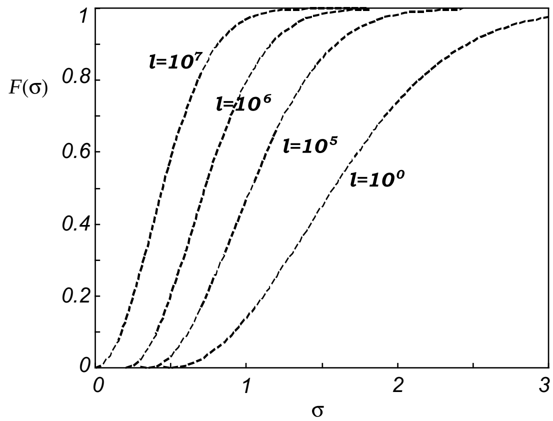

Fig. 7. Cumulative probability as a function of the stress σ and of the cycles l.

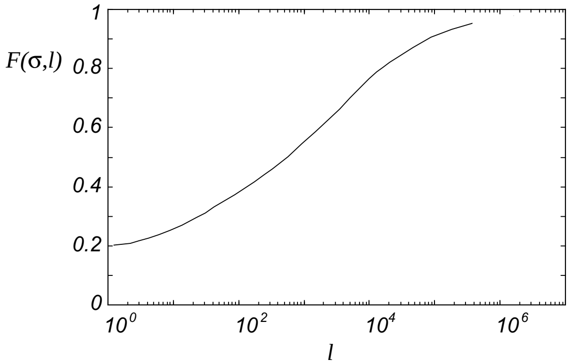

Fig. 8. Cumulative probability of occurrence, as a function of the number of cycles. $F(\sigma, 10^l) = F(\sigma, 10^0)[1 - k_1 \exp\{-k_2(\ln(l))^P\}]$.

6. Determination of the parameters in cyclic stress

When we consider cyclic stress the cumulative probability $F(\sigma, l)$ may be written as follows:

$$F(\sigma, l) = \left[1 - \exp\left\{-\left(\frac{\sigma - \sigma_L}{\sigma_0}\right)^m\right\}\right][1 - k_1 \exp\{-k_2(\ln l)^p\}] \tag{13}$$

in which m, σ_0 and σ_L are the known Weibull's parameters (Kittl and Díaz, 1988; Kittl et al., 2001) and k_1, k_2 and p are other parameters that may describe the process of fatigue due to cyclic stress (Kanninen and Popelar, 1985; Suresh, 1991). We can determine parameters m^0, σ_0^0, σ_L^0, k_1^0, k_2^0 and p^0 as a first approximation to true parameters m, σ_0, σ_L, k_1, k_2 and p, and in this case only six points are enough to determine them. In order to get a better approximation, the following expressions can be minimized:

$$\begin{aligned} &\text{Min} \sum_{i,j} [F_{ij} - F(\sigma, l)]^2 \\ &F_{ij} = \frac{n_{ij}}{N}; \quad \sigma_i \leqslant \sigma; \quad l_j \leqslant l \\ &F(\sigma, l) = F(\sigma, l; m, \sigma_0, \sigma_L, k_1, k_2, p) \end{aligned} \tag{14}$$

where F_{ij} is the observed cumulative probability, N is the number of samples tested and n_{ij} is the number of samples subjected to σ_i with l_j number of cycles. If we made a differential expansion to $F(\sigma, l)$ the following expression is obtained:

$$\begin{aligned} F(\sigma,l) &= F(\sigma,l;m=m^0+\partial m,\sigma_0=\sigma_0^0+\partial\sigma_0,\sigma_\mathrm{L}=\sigma_\mathrm{L}^0+\partial\sigma_\mathrm{L},k_1=k_1^0+\partial k_1,k_2=k_2^0+\partial k_2,\\ &\qquad p=p^0+\partial p^0)\\ &= F(\sigma,l;m^0,\sigma_0^0,\sigma_\mathrm{L}^0,k_1^0,k_2^0,p^0)+\frac{\partial F(\sigma,l;0)}{\partial m}\partial m+\frac{\partial F(\sigma,l;0)}{\partial\sigma_0}\partial\sigma_0+\frac{\partial F(\sigma,l;0)}{\partial\sigma_\mathrm{L}}\partial\sigma_\mathrm{L}\\ &\quad+\frac{\partial F(\sigma,l;0)}{\partial k_1}\partial k_1+\frac{\partial F(\sigma,l;0)}{\partial k_2}\partial k_2+\frac{\partial F(\sigma,l;0)}{\partial p}\partial p \end{aligned} \tag{15}$$

In Eq. (15) ∂m, $\partial\sigma_0$, $\partial\sigma_\mathrm{L}$, ∂k_1, ∂k_2 and ∂p can be obtained in such a way that Eq. (14) is minimized by using the common least squares method. Finally:

$$\begin{aligned} m &= m^0+\partial m\\ \sigma_0 &= \sigma_0^0+\partial\sigma_0\\ \sigma_\mathrm{L} &= \sigma_\mathrm{L}^0+\partial\sigma_\mathrm{L}\\ k_1 &= k_1^0+\partial k_1\\ k_2 &= k_2^0+\partial k_2\\ p &= p^0+\partial p \end{aligned} \tag{16}$$

7. Dispersion of the parameters in cyclic stress

In order to obtain the dispersion of the parameters in the case of the material being subjected to cyclic stress we made a numerical simulation with a process similar to that described for materials subjected to gradual loads. The procedure used here was the following: cumulative probability F has to be a random number and is obtained from a random number generation function, letting $0 \leqslant \lambda_{ij} \leqslant 1$ be; this number may be transformed into a product $\lambda = \lambda_i\lambda_j$. We must randomly attribute λ_i to the first part of Eq. (13) or to the second part, $\lambda_{ij} \leqslant \lambda_i \leqslant 1$, then the second is $\lambda_j = \lambda/\lambda_i$. We obtain σ_i from λ_i and l_j from λ_j. Once N pairs (σ_i, l_j) are obtained, we can determine F_{ij}, according to Eq. (14) and the mean values of parameters m, σ_0, σ_L, k_1, k_2 and p are obtained for each one of those numerical simulations as well as their Δm, $\Delta\sigma_0$, $\Delta\sigma_\mathrm{L}$, Δk_1, Δk_2 and Δp. According to Fisher's matrix (Kittl and Díaz, 1988) it may be verified that:

$$\begin{aligned} \Delta m &= \frac{1}{\sqrt{N}}f(m,\sigma_0,\sigma_\mathrm{L},k_1,k_2,p); \quad & \Delta k_1 &= \frac{1}{\sqrt{N}}s(m,\sigma_0,\sigma_\mathrm{L},k_1,k_2,p)\\ \Delta\sigma_0 &= \frac{1}{\sqrt{N}}g(m,\sigma_0,\sigma_\mathrm{L},k_1,k_2,p); \quad & \Delta k_2 &= \frac{1}{\sqrt{N}}t(m,\sigma_0,\sigma_\mathrm{L},k_1,k_2,p)\\ \Delta\sigma_\mathrm{L} &= \frac{1}{\sqrt{N}}h(m,\sigma_0,\sigma_\mathrm{L},k_1,k_2,p); \quad & \Delta p &= \frac{1}{\sqrt{N}}u(m,\sigma_0,\sigma_\mathrm{L},k_1,k_2,p) \end{aligned} \tag{17}$$

where functions f, g, h, s, t and u depend only on m, σ_0, σ_L, k_1, k_2 and p, and they must be deduced from a particular case.

8. Design and quality control in cyclic stress

For some material subjected to cyclic stress, if cumulative probability F and its tolerance δF are fixed to estimate the resistance of the material to fatigue process, the variation of the δm, $\delta\sigma_0$, $\delta\sigma_L$, δk_1, δk_2 and δl produced by δF can be obtained according to the following equations:

$$\begin{aligned}
F &= \left[1-\exp\left\{-\left(\frac{\sigma-\sigma_L}{\sigma_0}\right)^m\right\}\right][1-k_1\exp\{-k_2(\ln l)^p\}]\\
F+\delta F &= \left[1-\exp\left\{-\left(\frac{\sigma-\sigma_L}{\sigma_0}\right)^{m+\delta m}\right\}\right][1-k_1\exp\{-k_2(\ln l)^p\}]\\
F+\delta F &= \left[1-\exp\left\{-\left(\frac{\sigma-\sigma_L}{\sigma_0+\delta\sigma_0}\right)^m\right\}\right][1-k_1\exp\{-k_2(\ln l)^p\}]\\
F+\delta F &= \left[1-\exp\left\{-\left(\frac{\sigma-(\sigma_L+\delta\sigma_L)}{\sigma_0}\right)^m\right\}\right][1-k_1\exp\{-k_2(\ln l)^p\}]\\
F+\delta F &= \left[1-\exp\left\{-\left(\frac{\sigma-\sigma_L}{\sigma_0}\right)^m\right\}\right][1-(k_1+\delta k_1)\exp\{-k_2(\ln l)^p\}]\\
F+\delta F &= \left[1-\exp\left\{-\left(\frac{\sigma-\sigma_L}{\sigma_0}\right)^m\right\}\right][1-k_1\exp\{-(k_2+\delta k_2)(\ln l)^p\}]\\
F+\delta F &= \left[1-\exp\left\{-\left(\frac{\sigma-\sigma_L}{\sigma_0}\right)^m\right\}\right][1-k_1\exp\{-k_2(\ln l)^{p+\delta p}\}]
\end{aligned} \tag{18}$$

which can determine δm, $\delta\sigma_0$, $\delta\sigma_L$, δk_1, δk_2 and δp. When we make, $\Delta m = \mu\delta m$, $\Delta\sigma_0 = \mu\delta\sigma_0$, $\Delta\sigma_L = \mu\delta\sigma_L$, $\Delta k_1 = \mu\delta k_1$, $\Delta k_2 = \mu\delta k_2$ and $\Delta p = \mu\delta p$, we obtain:

$$\begin{aligned}
&Nm = \mu^2\frac{f^2}{\Delta m^2};\quad N\sigma_0 = \mu^2\frac{g^2}{\Delta\sigma_0^2};\quad N\sigma_L = \mu^2\frac{h^2}{\Delta\sigma_L^2}\\
&Nk_1 = \mu^2\frac{s^2}{\Delta k_1^2};\quad Nk_2 = \mu^2\frac{t^2}{\Delta k_2^2};\quad Np = \mu^2\frac{u^2}{\Delta p^2}
\end{aligned} \tag{19}$$

When $\mu = 1$ then we have $\Delta m = \delta m$, $\Delta\sigma_0 = \delta\sigma_0$, $\Delta\sigma_L = \delta\sigma_L$, $\Delta k_1 = \delta k_1$, $\Delta k_2 = \delta k_2$ and $\Delta p = \delta p$, the reliability is approximately equal to 70%; when $\mu = 2$ the reliability is approximately equal to 90%, such reliabilities are obtained from larger of the Nm, $N\sigma_0$, $N\sigma_L$, Nk_1, Nk_2 and Np values.

From the quality control point of view, in the case that Nm, $N\sigma_0$, $N\sigma_L$, Nk_1, Nk_2 and Np are such that the largest of them is larger than the number of experimental data used, N, the number of test samples that must be made corresponds to the largest number, and all the parameters and σ_C for the given cumulative probability F must be determined again. As the dispersion of the parameters must be the appropriate to get a given percentage of them in accordance with admissible tolerance δF, the stress of design σ_C and the fashion in which the material used must be controlled are thus defined. If $\sigma_C = \sigma_C(l)$ is not constant but a function of l, the computations would be similar.

9. The case of a lower limit stress σ_L and an upper limit stress σ_S

The way in which both σ_L and σ_S are null means that there is not an upper limit for the size of the defects producing the failure, nor a lower limit. The upper limit is reflected in σ_L and the lower limit in σ_S. The three cases are shown in Figs. 9 and 10.

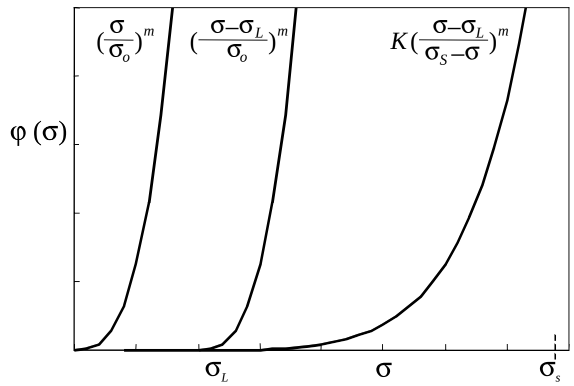

Fig. 9. The three cases of function $\varphi(\sigma)$ where it is equal to: $(\sigma/\sigma_0)^m$, $(\sigma - \sigma_L/\sigma_0)^m$ and $(\sigma - \sigma_L/\sigma_S - \sigma)^m K$.

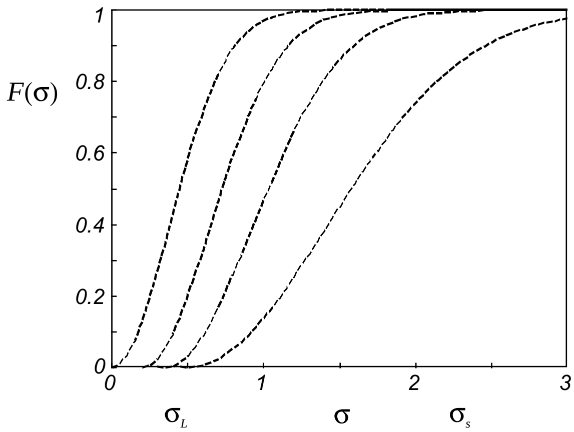

Fig. 10. The three cases of function $\varphi(\sigma)$ in $F(\sigma) = 1 - \exp[-\varphi(\sigma)]$, where $\varphi(\sigma)$ is equal to: $(\sigma/\sigma_0)^m$, $(\sigma - \sigma_L/\sigma_0)^m$ and $(\sigma - \sigma_L/\sigma_S - \sigma)^m K$. Functions $F(\sigma)$ are tangent in the points $\sigma = 0$, $\sigma = \sigma_S$, $\sigma = \sigma_L$.

The case we want to study now is the case of Kies–Kittl's cumulative probability function (Díaz and Morales, 1988)

$$F = 1 - \exp\left\{ - \left(\frac{\sigma - \sigma_L}{\sigma_S - \sigma} \right)^m K \right\} \tag{20}$$

where σ_S is the limit stress above which there is always failure, K is a parameter named Kittl's constant. From Eq. (20) we deduce (Kittl, 2002):

$$\ln \xi(\sigma) = \ln \left[\ln \left(\frac{1}{1 - F} \right) \right] = \ln K + m \ln(\sigma - \sigma_L) - m \ln(\sigma_S - \sigma) \tag{21}$$

Drawing the graph representing Eq. (21) we obtain Fig. 11.

In order to find parameters σ_L, σ_S, m and K that best fit the experimental data, let us calculate the tangent in the midpoint of σ_L and σ_S, $\sigma_m = (\sigma_L + \sigma_S)/2$:

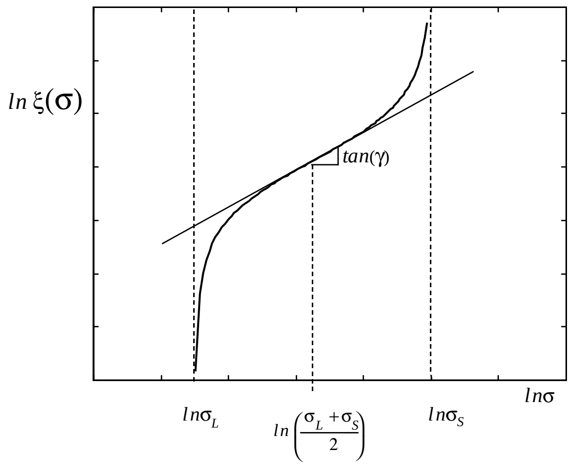

Fig. 11. Diagram of the Kies–Kittl function. The asymptotes of $\ln\xi(\sigma)$ are $\ln\sigma_{\mathrm{L}}$ and $\ln\sigma_{\mathrm{S}}$.

$$\frac{\mathrm{d}\ln\xi(\sigma)}{\mathrm{d}\ln\sigma}=\frac{\frac{\mathrm{d}\ln\xi(\sigma)}{\mathrm{d}\sigma}}{\frac{\mathrm{d}\ln\sigma}{d\sigma}}=m\frac{\sigma}{\sigma-\sigma_{\mathrm{L}}}+m\frac{\sigma}{\sigma_{\mathrm{S}}-\sigma} \tag{22}$$

hence:

$$\left[\frac{\mathrm{d}\ln\xi(\sigma)}{\mathrm{d}\ln\sigma}\right]_{\sigma=(\sigma_{\mathrm{L}}+\sigma_{\mathrm{S}})/2}=2m\frac{\sigma_{\mathrm{S}}+\sigma_{\mathrm{L}}}{\sigma_{\mathrm{S}}-\sigma_{\mathrm{L}}} \tag{23}$$

Therefore, if in the vicinity of point $\ln(\frac{\sigma_{\mathrm{L}}+\sigma_{\mathrm{S}}}{2})$ we calculate the tangent to the curve $\mathrm{tg}\gamma$ we shall have:

$$m=\frac{1}{2}\frac{\sigma_{\mathrm{S}}-\sigma_{\mathrm{L}}}{\sigma_{\mathrm{S}}+\sigma_{\mathrm{L}}}\left[\frac{\mathrm{d}\ln\xi(\sigma)}{\mathrm{d}\ln\sigma}\right]_{\sigma=(\sigma_{\mathrm{L}}+\sigma_{\mathrm{S}})/2} \tag{24}$$

In the real calculation procedure, from the engineering point of view, the values of σ_{L} and σ_{S} are estimated from Fig. 11, where the experimental values have been represented, and then, at point $\ln\frac{\sigma_{\mathrm{L}}+\sigma_{\mathrm{S}}}{2}$, the tangent to the curve is estimated with the points of the vicinity, drawing a parabola of third or fourth degree. The value of K is determined by using the following equation:

$$\sum_{i=1}^{N}\ln\xi(\sigma)=N\ln K+m\sum_{i=1}^{N}(\sigma_i-\sigma_{\mathrm{L}})+m\sum_{i=1}^{N}\ln(\sigma_{\mathrm{S}}-\sigma_i) \tag{25}$$

where N is the number of experimental values of stress σ_i. Then we have approximate values for σ_{L}, σ_{S}, m and K. Now, if we calculate, for different values of σ_{L}, a series of values of σ_{S}, that will provide the corresponding m and K and if we calculate the corresponding chi-square value, χ^2, we can finally obtain a minimum chi-square value, χ^2, that will provide the values of the parameters with an approximate value as large as we want to. In symbols,

$$\{\sigma_{i\mathrm{L}}\}\leftrightarrow\{\sigma_{j-1,S};\sigma_{j,S};\sigma_{j+1,S};\ldots\}\rightarrow\{m_{ij},K_{ij}\}\quad i=1,\ldots,N;\;\; j=1,\ldots,M \tag{26}$$

where M is an integer number. The numerical iteration process, applying the method explained here for different values of σ_{L}, is shown in Fig. 12.

Changing σ_{L}, $\{\sigma_{i,\mathrm{L}}\}$ a set of curves is obtained whose minimum gives us a first approximation to the parameters. In Fig. 13 we show how such process works.

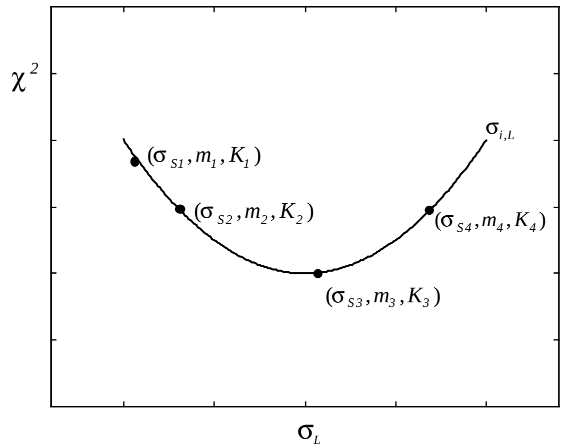

Fig. 12. Variation of chi-square, χ^2 for an estimated value of σ_L, $\sigma_{i,L}$ and several values of σ_S.

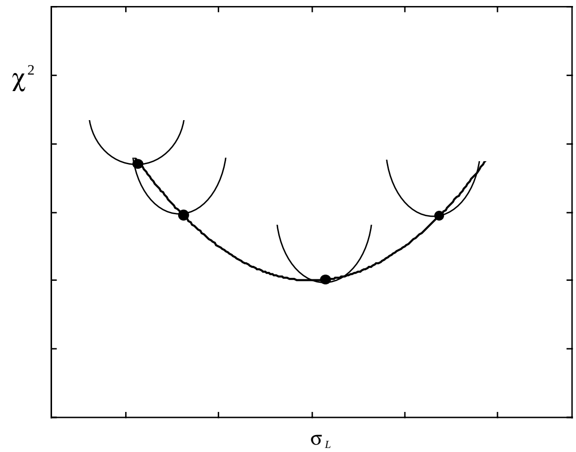

Fig. 13. Curves of chi-square, χ^2 versus different σ_{Li}. The minimum of chi-square, χ^2 provides σ_L as a first approximation.

10. Conclusions

Using a numerical simulation method we determined the number of samples to be tested needed to characterize the design and quality control for some material subjected to traction for a given mechanical property as yield point, elastic limit stress, fracture strength, etc. Such number of samples must satisfy a fixed cumulative probability with a fixed admissible tolerance for them. The method was employed for materials subjected to graduals stress and cyclic stress. The method was shown for AISI 1020 steel. For the case of gradual stress by means of random number generation function 30 stress values were simulated and after 100 iterations Weibull's parameters dispersions were obtained. Then with both fixed cumulative probability and the admissible tolerance for them and with the design stress the variations of the Weibull's parameters were obtained. After that, employing Fisher's matrix, which allows for a relation between the number of samples tested with the dispersion of Weibull's parameters, the number of samples to be tested were obtained. This procedure allows us to expect approximately 70% of the stress values to be within the range of acceptance. In addition, in accordance to design stress value the error would not to be large and the design will be safer if Weibull's parameter σ_L takes zero value. A similar procedure was used for a cyclic stress case where the random numbers generated were transformed into a product to solve the determination of Weibull's parameters and the fatigue parameters from the joint cumulative probability function separately. In the case of the cumulative probability function having an upper and lower limit stress the parameters were estimated by numerical simulation approximation until a minimum chi-square was obtained. Finally, the method explained here can be used, for example, as a quality control procedure to accept or reject a portion of manufactured series material that follows a Weibull cumulative probability function for some mechanical property with random behavior.

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